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GEODESIC SYMMETRIES IN SPACES WITH SPECIAL CURVATURE TENSORS

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In [3], the authors initiated a study of Riemannian manifolds whose local geodesic symmetries are divergence-preserving (volume-preserving up to sign). We found an infinite sequence of necessary conditions on the curvature tensor, which are sufficient in the analytic case. These results extend to pseudo-Riemannian manifolds with no essential change in proof.

In this paper, we show that the necessary conditions are satisfied in a broad class of spaces defined by imposing a simple algebraic condition on the first covariant derivative of the curvature tensor. This class includes naturally reductive pseudo-Riemannian spaces. We also consider a family of examples, constructed by N. R. Wallach [9] in another context, which shows that there exist reductive Riemannian homogeneous spaces, whose geodesic symmetries fail to be divergence-preserving, and others which satisfy our first necessary condition but not the second.

In spaces which satisfy our algebraic condition, the necessary conditions reduce to verifying an infinite sequence of combinatorial identities involving sums over k indices, $k = 1, 2, \cdots$. The authors succeeded in verifying these for $k \leq 3$, and wish to thank A. Poritz and, especially, D. Slater for assistance in running computer programs associated with the proof of the case $k = 3$. A proof for general k has now been provided by R. T. Bumby [2]. A purely algebraic corollary guarantees the vanishing of the trace of certain recursively defined compositions of two endomorphisms of a finite dimensional vector space. The authors also thank N. R. Wallach for helpful conversations.

1. Preliminaries

Let X denote a nonzero vector at a point 0 of a C^∞ pseudo-Riemanian manifold M, and define an endomorphism Π of the tangent space $T_0(M)$ by

$$(1) \qquad \Pi(Y) = -R(Y,X)X \,, \qquad Y \in T_0(M) \,,$$

where R denotes the curvature of the canonical torsionless metric connection ∇. Let $\nabla^0_X\Pi = \Pi$, and define $\nabla^i_X\Pi$, $i = 1, 2, \cdots$, by first extending X to the

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velocity vector field along the geodesic through 0 determined by X and then extending Π in accordance with (1). For later reference, we note that

$$(2)\qquad (\nabla_X \Pi)Y = -(\nabla_X R)(Y,X)X \,.$$

Finally, define endomorphisms P^r, $r = 2, 3, \cdots$, of $T_0(M)$ by the recurrence formula

$$(3)\qquad (r+1)P^r = r(r-1)\nabla_X^{r-2}\Pi - \sum_{q=2}^{r-2} \binom{r}{q} P^q \circ P^{r-q}$$

of Ledger [8]. Our necessary conditions [3] are that

$$(4)\qquad \operatorname{trace} P^r = 0 \,, \qquad r = 3, 5, 7, \cdots \,,$$

for all choices of 0 and X.

From the recurrence formula we find that

$$(5)\qquad P^r = \sum c^r_{i_1\cdots i_k} \nabla_X^{i_1}\Pi \circ \cdots \circ \nabla_X^{i_k}\Pi \,,$$

where the (absolute) constants $c^r_{i_1\cdots i_k}$, defined only for $r = i_1 + \cdots + i_k + 2k \geq 2$, $i_j \geq 0$, $1 \leq k \leq [r/2]$, are given by

$$(6)\qquad c^r_{i_1} = c^r_{r-2} = \frac{r(r-1)}{r+1}$$

for $k = 1$ and, for $k > 1$, by

$$(7)\qquad c^r_{i_1\cdots i_k} = -\frac{1}{r+1} \sum_{j=1}^{k-1} \binom{r}{q_j} c^{q_j}_{i_1\cdots i_j} c^{r-q_j}_{i_{j+1}\cdots i_k}$$

with $q_j = i_1 + \cdots + i_j + 2j$.

Details of these results are given in [3].

2. Special curvature tensor

A C^∞ pseudo-Riemannian manifold has a *special curvature tensor* in the sense of the title of this paper, if it carries a C^∞ tensor T of type (1,2) (written as $T_X Y = T(X, Y)$ where T_X is viewed as an endomorphism of the tangent space) such that

$$(8)\qquad (\nabla_X R)(Y,X)X = T_X R(Y,X)X - R(T_X Y, X)X \,,$$

$$(9)\qquad (\nabla_X T)_X = 0$$

for all vectors X and Y.

For these spaces, the $\nabla_X^i \Pi$ can be computed algebraically by

$$\nabla_X^i \Pi = (\mathrm{ad}\, T_X)^i \Pi \ , \tag{10}$$

where $(\mathrm{ad}\, T_X)\Sigma$, for any endomorphism Σ of the tangent space, denotes the endomorphism $[T_X, \Sigma] = T_X\Sigma - \Sigma T_X$.

In fact, for $i = 1$, (10) is the same as (8), by (2). For $i > 1$, the proof is by induction, using the following lemma which follows from (9).

Lemma 2.1. *Let Σ be a differentiable family of endomorphisms of tangent spaces (equivalently a tensor of type (1,1)) defined along the geodesic through 0 determined by X. Then*

$$\nabla_X[T_X, \Sigma] = [T_X, \nabla_X\Sigma] \ , \tag{11}$$

where, on the left, T_X is taken along the geodesic through 0 determined by X.

Note that an alternate version of (11) is

$$\nabla_X((\mathrm{ad}\, T_X)\Sigma) = (\mathrm{ad}\, T_X)\nabla_X\Sigma \ . \tag{11'}$$

Proof. Along a geodesic, we have $\nabla_X X = 0$. Then (9) gives

$$\nabla_X(T_X Z) = T_X \nabla_X Z \tag{9'}$$

for arbitrary vector fields Z. (9′) is used with $Z = \Sigma Y$ and $Z = \nabla_X Y$ (arbitrary Y) to identify the first and fourth terms, respectively, in computing

$$\begin{aligned}
(\nabla_X[T_X, \Sigma])Y &= \nabla_X([T_X, \Sigma]Y) - [T_X, \Sigma]\nabla_X Y \\
&= \nabla_X(T_X\Sigma Y) - \nabla_X(\Sigma T_X Y) - T_X\Sigma\nabla_X Y + \Sigma T_X \nabla_X Y \ , \\
[T_X, \nabla_X\Sigma]Y &= T_X(\nabla_X\Sigma)Y - (\nabla_X\Sigma)T_X Y \\
&= T_X\nabla_X(\Sigma Y) - T_X\Sigma\nabla_X Y - \nabla_X(\Sigma T_X Y) + \Sigma\nabla_X(T_X Y) \ .
\end{aligned}$$

Proof of induction step for (10), using (11′):

$$\begin{aligned}
\nabla_X^{i+1}\Pi &= \nabla_X(\nabla_X^i \Pi) = \nabla_X((\mathrm{ad}\, T_X)\nabla_X^{i-1}\Pi) \\
&= (\mathrm{ad}\, T_X)\nabla_X(\nabla_X^{i-1}\Pi) = (\mathrm{ad}\, T_X)(\mathrm{ad}\, T_X^i)\Pi = (\mathrm{ad}\, T_X)^{i+1}\Pi \ .
\end{aligned}$$

For spaces which satisfy (10), a consequence of (8) and (9), the necessary conditions (4) that geodesic symmetries be divergence-preserving follow from certain combinatorial identities.

First note that

$$(\mathrm{ad}\, T_X)^i \Pi = \sum_{p=0}^{i} (-1)^{i-p} \binom{i}{p} T_X^p \Pi T_X^{i-p} \ . \tag{12}$$

Using (10) and (12), we can write (5) as

$$(13)\qquad P^r = \sum c^r_{i_1\cdots i_k}(-1)^\lambda\binom{i_1}{p_1}\cdots\binom{i_k}{p_k}T_X^{p_1}\Pi T_X^{i_1-p_1}\cdots T_X^{p_k}\Pi T_X^{i_k-p_k}\ ,$$

where the summation is extended further over all p_j, $0 \le p_j \le i_j$, and $\lambda = i_1 + \cdots + i_k - (p_1 + \cdots + p_k)$.

In computing trace P^r we use the facts that trace is linear and invariant under cyclic permutation of endomorphisms, and then group terms, within a common value of $r - 2k$, when the endomorphism terms are cyclic permutations of one another, and obtain

$$(14)\qquad \begin{aligned}&(r+1)\ \mathrm{trace}\ P^r\\ &= (-1)^r r!\sum(-1)^{k-1}\sum\frac{1}{\sigma_1!\cdots\sigma_k!}\frac{\mathrm{per}\,(\sigma)}{k}F(\sigma_1,\cdots,\sigma_k)\\ &\qquad\cdot\mathrm{trace}\,(T_X^{\sigma_1}\Pi T_X^{\sigma_2}\Pi\cdots T_X^{\sigma_k}\Pi)\ .\end{aligned}$$

The final summation is over equivalence classes $\sigma = (\sigma_1, \cdots, \sigma_k)$ of nonnegative integers whose sum is $r - 2k$ (equivalent if one sequence is a cyclic permutation of another). Also, per (σ) denotes the smallest positive integer m such that $\sigma_j = \sigma_{j+m}$ for all j satisfying $1 \le j \le k - m$; i.e., m is the number of distinct k-tuples in the equivalence class. In obtaining (14) from (13), we have introduced the notation

$$(15)\qquad \sigma_1 = i_k - p_k + p_1\ ,\quad \sigma_j = i_{j-1} - p_{j-1} + p_j\ ,\quad 1 < j \le k\ ,$$

used the fact that

$$(16)\qquad \sigma_1 + \cdots + \sigma_k = i_1 + \cdots + i_k = r - 2k\ ,$$

and introduced functions $F(\sigma_1, \cdots, \sigma_k)$ which will be described below.

Theorems 2.1 and 2.2 (below) will follow from a proof that the combinatorial functions $F(\sigma_1, \cdots, \sigma_k)$ satisfy

$$(17)\qquad F(\sigma_1, \cdots, \sigma_k) = 0 \qquad \text{if } \sigma_1 + \cdots + \sigma_k \text{ is odd}\ .$$

For $k = 1$, with the value of the coefficient $c^r_{i_1}$ in (13) taken from (6), we find

$$F(\sigma_1) = \sum_{p_1=0}^{\sigma_1}(-1)^{p_1}\binom{\sigma_1}{p_1}\ ,$$

which is well known to vanish for any positive integer σ_1.

For general k, it is convenient to introduce new quantities

$$b(i_1, \cdots, i_k) \\ = (-1)^{k-1} i_1! \cdots i_k! \, c^r_{i_1 \cdots i_k} \frac{i_1 + \cdots + i_k + 2k + 1}{(i_1 + \cdots + i_k + 2k)!} . \tag{18}$$

Then (6) gives

$$b(i_1) = 1 , \tag{19}$$

and (7) gives the recursion formula

$$b(i_1, \cdots, i_k) \\ = \sum_{j=1}^{k-1} \frac{b(i_1, \cdots, i_j)}{i_1 + \cdots + i_j + 2j + 1} \frac{b(i_{j+1}, \cdots, i_k)}{i_{j+1} + \cdots + i_k + 2k - 2j + 1} \tag{20}$$

if $k > 1$. Finally, define

$$B(i_1, \cdots, i_k) = b(i_1, \cdots, i_k) + b(i_2, \cdots, i_k, i_1) \\ + \cdots + b(i_k, i_1, \cdots, i_{k-1}) . \tag{21}$$

Then

$$F(\sigma_1, \cdots, \sigma_k) = \sum (-1)^{p_1 + \cdots + p_k} \binom{\sigma_1}{p_1} \cdots \binom{\sigma_k}{p_k} B(i_1, \cdots, i_k) , \tag{22}$$

where the summation is over $0 \leq p_j \leq \sigma_j$, $1 \leq j \leq k$ with i's determined from (15). Note that the B's, and therefore the F's, are invariant under cyclic permutation of the arguments.

To derive (22), we start from (13), replace c's in terms of b's from (18) and write out all combinatorial coefficients in terms of factorials; then multiply through by $(\sigma_1! \cdots \sigma_k!)$ and divide by same, always using (15) and (16); finally regroup the factorials into appropriate combinatorial coefficients, take trace and compare with (14).

The authors were able to prove (17) for $k = 2$ (trivially) and for $k = 3$ (laboriously). A proof of (17) for general k has been found by R. T. Bumby [2]. His paper takes (19), (20), (21), (22), together with the notation (15), as definitions, and is independent of the differential geometric aspects of the problem.

From (17), proved in [2], we obtain

Theorem 2.1. *Let M be a real analytic pseudo-Riemannian manifold which has, in a neighborhood of each point, a C^∞ tensor field T of type* (1,2) *satisfying* (8) *and* (9). *Then the geodesic symmetries are locally divergence-preserving.*

The proof obviously extends to the (possibly) larger class of manifolds for which (10) holds locally, and to purely algebraic situations. We have

Theorem 2.2. *Let T and Π be endomorphisms of a finite-dimentional vector space. Define additional endomorphisms P^r, $r = 2, 3, \cdots$, recursively by*

$$(r + 1)P^r = r(r - 1)(\mathrm{ad}\, T)^{r-2}\Pi - \sum_{q=2}^{r-2} \binom{r}{q} P^q \circ P^{r-q} ,$$

with $(\mathrm{ad}\, T)^0\Pi = \Pi$. *Then trace $P^r = 0$ if r is odd.*

This result can also be extended to operators on algebras more general than trace whenever the operator is linear and invariant under cyclic permutation of a product.

3. Reductive pseudo-Riemannian homogeneous spaces

For naturally reductive Riemannian homogeneous spaces, the existence of a T satisfying conditions stronger than (8) and (9) follows from the work of Ambrose-Singer [1] and of Kostant [7]. An explicit formula for T (valid also in the pseudo-Riemannian case) will follow from a special case of the material given below for reductive spaces. Since these spaces have a subordinate real analytic structure, Theorem 2.1 yields

Theorem 3.1. *In a naturally reductive pseudo-Riemannian homogeneous space, the local geodesic symmetries are divergence-preserving.*

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and K a closed subgroup with Lie algebra $\mathfrak{k}$. Let $M = G/K$, and assume that G acts effectively on M. The canonical projection $\pi: G \to G/K = M$ takes $e \in G$ into $0 = \pi(e) \in M$. For any $X \in \mathfrak{g}$, let X^* denote the global vector field induced on M by the (left) action of the one-parameter family $\exp tX$. Then $(X^*)_0 = \pi_* X$ and $[X^*, Y^*] = -[X, Y]^*$. Identities relating G-invariant tensors hold at all points of M if they can be verified at 0.

Now assume we have a vector space direct sum decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ such that $\mathrm{ad}\,(K)\mathfrak{p} \subset \mathfrak{p}$, i.e., M is reductive. This condition ensures that the definitions and identities below can be translated into Lie algebra statements when π_* is used to identify $\mathfrak{p}$ with $T_0(M)$. For $X \in \mathfrak{g}$, we write $X = X_{\mathfrak{k}} + X_{\mathfrak{p}}$ to identify components under the decomposition. The canonical G-invariant connection $\tilde{\nabla}$ is defined by

$$(\tilde{\nabla}_{X^*} Y^*)_0 = -[X, Y]_{\mathfrak{p}} \qquad \text{for } X, Y \in \mathfrak{p} ,$$

and the natural torsionless connection $\bar{\nabla}$ by

$$(\bar{\nabla}_{X^*} Y^*)_0 = -\tfrac{1}{2}[X, Y]_{\mathfrak{p}} \qquad \text{for } X, Y \in \mathfrak{p} .$$

Now assume given a G-invariant pseudo-Riemannian structure on M, and let ∇ denote the associated torsionless metric connection. These structures are in one-one correspondence with $(\operatorname{ad} K)$-invariant nondegenerate symmetric bilinear forms B on $\mathfrak{p}$, so we have a given B. Define a G-invariant tensor field U on M by

$$U(X^*, Y^*) = \nabla_{X^*}Y^* - \bar{\nabla}_{X^*}Y^* .$$

Then U is symmetric since both ∇ and $\bar{\nabla}$ are torsionless. The pull-back to $\mathfrak{p}$ of U at 0 is characterized by

$$2B(U(X, Y), Z) = B(X, [Z, Y]_{\mathfrak{p}}) + B([Z, X]_{\mathfrak{p}}, Y) \tag{23}$$

for X, Y, $Z \in \mathfrak{p}$, with $U \equiv 0$ if and only if the structure is naturally reductive (with respect to the choice of $\mathfrak{p}$). These results can be found in Chapter 10 of [6], together with the proposition that G-invariant tensor fields are covariant-constant with respect to $\tilde{\nabla}$.

Finally, define a G-invariant tensor field T on M by

$$T_{X^*}Y^* = \nabla_{X^*}Y^* - \tilde{\nabla}_{X^*}Y^* ,$$

and note that

$$T_X Y = U(X, Y) + \tfrac{1}{2}[X, Y]_{\mathfrak{p}} \tag{24}$$

for X, $Y \in \mathfrak{p} = T_0(M)$. Let R denote the curvature tensor of the torsionless metric connection ∇. Then $\tilde{\nabla}R = 0$ and $\tilde{\nabla}T = 0$ are equivalent to

$$(\nabla_Z R)(Y, X) = -R(T_Z Y, X) + R(T_Z X, Y) + [T_Z, R(Y, X)] , \tag{25}$$

$$(\nabla_Y T)_X = -T_{T_Y X} + [T_Y, T_X] \tag{26}$$

for all X, Y, $Z \in \mathfrak{p} = T_0(M)$.

In the naturally reductive case, $U \equiv 0$ is equivalent to $T_X X = 0$ by (24) and polarization, so that (25) and (26) imply (8) and (9) as required for Theorem 3.1.

For reductive spaces, (25) and (26) can be used to reformulate the conditions (4) as Lie algebra computations. The theorem below covers only the first two conditions, but is useful in checking concrete examples as in § 4.

Theorem 3.2. *Let M be a reductive pseudo-Riemannian space such that local geodesic symmetries are divergence-preserving. Let S denote the Ricci curvature tensor of the torsionless metric connection. Then*

$$S(X, U(X, X)) = \operatorname{trace}\{Y \to R(Y, X)U(X, X)\} = 0 , \tag{27}$$

$$\operatorname{trace}\{Y \to R(R(Y, X)X, X)U(X, X)\} = 0 \tag{28}$$

for all $X \in \mathfrak{p}$.

Proof. For $r = 3$, we use (5), (6), (2), (25), and (1) to obtain

$$\frac{2}{3}P^3(Y) = [T_X, \Pi]Y - R(T_X X, Y)X + R(Y, X)T_X X .$$

Then

$$\text{trace}\, \frac{2}{3}P^3 = S(T_X X, X) + S(X, T_X X) = 2S(X, U(X, X))$$

since $T_X X = U(X, X)$ by (24).

For $r = 5$, given trace $P^3 = 0$ we reduce the condition trace $P^5 = 0$ to

$$\text{trace}\, \nabla_X \Pi^2 = \text{trace}\, (\nabla_X \Pi \circ \Pi + \Pi \circ \nabla_X \Pi) = 2\, \text{trace}\, (\nabla_X \Pi \circ \Pi) = 0$$

(cf. [3, p. 472]). To obtain (28) we express $(\nabla_X \Pi \circ \Pi)Y$ in terms of R and T, and then use the fact that

$$\text{trace}\, \{Y \to R(\Pi(Y), Z)W\}$$

is symmetric in the arguments Z and W, which can be proved easily by imitating the proof that the Ricci tensor S is symmetric.

In testing Theorem 3.2 on examples, we compute U from (23) and R at the point 0 from

Lemma 3.3. *For $X, Y, Z \in \mathfrak{p}$, we have*

$$\begin{aligned} R(X, Y)Z = & -[[X, Y]_{\mathfrak{k}}, Z] - \tfrac{1}{2}[[X, Y]_{\mathfrak{p}}, Z]_{\mathfrak{p}} - U([X, Y]_{\mathfrak{p}}, Z) \\ & + \tfrac{1}{4}[X, [Y, Z]_{\mathfrak{p}}]_{\mathfrak{p}} + \tfrac{1}{2}[X, U(Y, Z)]_{\mathfrak{p}} + U(X, U(Y, Z)) \\ & + \tfrac{1}{2}U(X, [Y, Z]_{\mathfrak{p}}) - \tfrac{1}{4}[Y, [X, Z]_{\mathfrak{p}}]_{\mathfrak{p}} - \tfrac{1}{2}[Y, U(X, Z)]_{\mathfrak{p}} \\ & - U(Y, U(X, Z)) - \tfrac{1}{2}U(Y, [X, Z]_{\mathfrak{p}}) . \end{aligned} \tag{29}$$

Proof. Use the formula expressing the difference of the curvature tensors of two connections in terms of the difference tensor T of the connections (cf. [4, p. 398], for example), also the well-known value at 0 of the curvature tensor for the canonical connection $\tilde{\nabla}$, the fact that $\tilde{\nabla} T = 0$, and finally (24).

4. A family of examples

Let $G = SU(3)$, and let K be the maximal toral subgroup. In the Lie algebra $\mathfrak{g} = \mathfrak{su}(3)$ of complex skew-hermitian 3×3 matrices with zero trace, $\mathfrak{k}$ is the subalgebra of diagonal matrices. Define $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ by

$$\mathfrak{p} = V_1 \oplus V_2 \oplus V_3 ,$$

where

$$V_1 = \left\{ \begin{bmatrix} 0 & z & 0 \\ -\bar{z} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad z \in \boldsymbol{C} \right\}, \qquad V_2 = \left\{ \begin{bmatrix} 0 & 0 & z \\ 0 & 0 & 0 \\ -\bar{z} & 0 & 0 \end{bmatrix}, \quad z \in \boldsymbol{C} \right\},$$

$$V_3 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & z \\ 0 & -\bar{z} & 0 \end{bmatrix}, \quad z \in \boldsymbol{C} \right\}.$$

Now let c_1, c_2, c_3 be nonzero real constants, and define a G-invariant pseudo-Riemannian structure on $M = G/K$ by

$$B(X, Y) = \begin{cases} 0\,, & \text{if } X \in V_i,\ Y \in V_j,\ i \neq j\,, \\ -c_i \operatorname{trace} XY\,, & \text{if } X,\ Y \in V_i,\ i = 1, 2, 3\,. \end{cases} \tag{30}$$

These spaces were introduced by N. R. Wallach [9] with nonnegative c_1, c_2, c_3, who also found

$$U(X, Y) = \begin{cases} 0\,, & \text{if } X, Y \in V_i,\ i = 1, 2, 3\,, \\ -\dfrac{c_i - c_j}{2c_k}[X, Y]\,, & \text{if } X \in V_i,\ Y \in V_j,\ i, j, k \text{ distinct}\,, \end{cases} \tag{31}$$

which follows from (23) and (30). In particular, the structure is naturally reductive if and only if $c_1 = c_2 = c_3$.

Define a basis $\{X_1, X_{\bar{1}}, X_2, X_{\bar{2}}, X_3, X_{\bar{3}}\}$ for $\mathfrak{p}$ by taking $z = 1$, i in V_1, $z = 1$, $-i$ in V_2, and $z = -1$, $-i$ in V_3. Then

$$[X_i, X_j] = -[X_{\bar{i}}, X_{\bar{j}}] = X_k\,, \qquad [X_{\bar{i}}, X_j] = [X_i, X_{\bar{j}}] = -X_{\bar{k}}\,,$$

if i, j, k is a cyclic permutation of 1, 2, 3. Define $K_i \in \mathfrak{k}$ by

$$[X_i, X_{\bar{i}}] = 2K_i\,, \qquad i = 1, 2, 3\,.$$

Then

$$[K_i, X_i] = 2X_{\bar{i}}\,, \quad [K_i, X_{\bar{i}}] = -2X_i\,, \qquad i = 1, 2, 3\,,$$
$$[K_i, X_j] = -X_{\bar{j}}\,, \quad [K_i, X_{\bar{j}}] = X_j\,, \qquad i \neq j\,.$$

The curvature tensor can be computed, with respect to this basis, from (29). The nontrivial cases are

$$R(X_i, X_{\bar{i}})X_i = -4X_{\bar{i}}\,, \qquad i = 1, 2, 3\,,$$

and, with i, j, k distinct,

$$\begin{aligned} R(X_i, X_{\bar{i}})X_j &= 2R(X_i, X_j)X_{\bar{i}} = -2R(X_{\bar{i}}, X_j)X_i \\ &= 2\{1 - \tfrac{1}{4}(c_i - c_j - c_k)^2/(c_j c_k)\}X_{\bar{j}}\,, \end{aligned}$$

$$
\begin{aligned}
R(X_i, X_j)X_i &= R(X_{\bar{i}}, X_j)X_{\bar{i}} \\
&= \left\{ \frac{(c_k - c_i)}{c_j} - \frac{(c_i - c_j - c_k)^2}{4c_j c_k} \right\} X_j
\end{aligned}
$$

together with the formulas obtained from those above reversing the roles of barred and unbarred indices on basis elements.

The only nontrivial terms of the Ricci tensor are

$$
S(X_i, X_i) = S(X_{\bar{i}}, X_{\bar{i}}) = (6c_j c_k + c_i^2 - c_j^2 - c_k^2)/(c_j c_k) , \tag{32}
$$

where i, j, k are distinct.

When (31) and (32) are used to compute (27), we find that our first necessary condition (27) is satisfied if and only if

$$
\frac{c_1 - c_2}{c_3} + \frac{c_2 - c_3}{c_1} + \frac{c_3 - c_1}{c_2} = 0 . \tag{33}
$$

The case where $c_1 = c_2 = 1$ and $c_3 = 2$ satisfies (27), but a lengthy argument (not given here) shows that (28) cannot be satisfied for all choices of X.

5. Appendix

N. R. Wallach has observed that the conclusion of Theorem 3.1 can be obtained for a broad class of naturally reductive pseudo-Riemannian homogeneous spaces by computing directly the action of the geodesic symmetry on the volume element. Wallach's argument is included here with his permission.

Let $M = G/K$ be a reductive homogeneous space, notation as in § 3, which carries a G-invariant volume element ω. Let U be a neighborhood of zero in $\mathfrak{p}$ such that $X \in U$ implies $-X \in U$ and also such that the map $\Psi(X) = \exp X \cdot K$ is a diffeomorphism of U onto an open neighborhood of 0 in M.

From the formula

$$
d \exp_X = d(L_{\exp X})_e \circ (I - e^{-\operatorname{ad} X})/(\operatorname{ad} X) , \qquad X \in \mathfrak{g} ,
$$

(cf. [5]), one obtains

$$
d\Psi_X = d(\exp X)_0 \circ d\Pi_e \circ (I - e^{-\operatorname{ad} X})/(\operatorname{ad} X) , \qquad X \in \mathfrak{p} ,
$$

where $\exp X$ is considered as a mapping of M. Let $\tilde{\omega} = (\Psi^* \omega)_0$ and take $\tilde{\omega}$ as a volume element on $\mathfrak{p}$, invariant under translation and invariant up to sign under the map $X \to -X$. If P denotes the projection $\mathfrak{g} \to \mathfrak{p}$, then one finds

$$
(\Psi^* \omega)_X = \det (P \circ (I - e^{-\operatorname{ad} X})/(\operatorname{ad} X)) \tilde{\omega}_X .
$$

Now specialize to M such that the pseudo-Riemannian naturally reductive homogeneous structure is induced by restricting to $\mathfrak{p}$ an (ad G)-invariant symmetric bilinear form B on $\mathfrak{g}$, which is nondegenerate on $\mathfrak{p}$ and makes $\mathfrak{p}$ and $\mathfrak{k}$ orthogonal (compare [6, II, p. 203]). Let

$$A(X) = P \circ (I - e^{\operatorname{ad} X})/(\operatorname{ad} X)|_{\mathfrak{p}} .$$

Choose an orthogonal basis $\{Z_1, \cdots, Z_n\}$ for $\mathfrak{p}$ so that $B(Z_i, Z_i) = \varepsilon_i = \pm 1$, and define $a_{ij}(X)$ by

$$A(X)Z_i = \sum_{j=1}^{n} a_{ij}(X)Z_j ,$$

that is,

$$a_{ij}(X) = \varepsilon_j B(A(X)Z_i, Z_j) .$$

From the series expansion of $(I - e^{-\operatorname{ad} X})/(\operatorname{ad} X)$, we compute

$$\begin{aligned} \varepsilon_j a_{ij}(X) &= \sum \frac{(-1)^k}{(k+1)!} B(P \circ (\operatorname{ad} X)^k Z_i, Z_j) \\ &= \sum \frac{(-1)^k}{(k+1)!} B(Z_i, (\operatorname{ad}(-X))^k Z_j) , \end{aligned}$$

where the last step depends upon the special properties of B assumed above. Then it follows that

$$\varepsilon_j a_{ij}(X) = \varepsilon_i a_{ji}(-X) , \tag{34}$$

and hence, a fortiori, that

$$\det A(X) = \det A(-X) .$$

The conclusion follows from the fact that the pullback by Ψ of the geodesic symmetry on M around 0 is just $X \to -X$.

For manifolds M which can carry a special metric as assumed above, the conclusion follows also for any other naturally reductive pseudo-Riemannian metric compatible with the given decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. This is because both metrics will have the same geodesic symmetry around 0 with volume elements differing by at most a multiplicative constant.

Added in proof. We can now prove (34) for all naturally reductive pseudo-Riemannian spaces. The proof depends on an inductive argument showing that a computation, similar to that just preceeding (34), is valid without assuming special properties of B.

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